PENETRATION FACTOR IN ALPHA-DECAY

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ABSTRACT. In this paper we have calculated the penetration probability of α-particle through a potential barrier by using the Lanczos method of solving the Schrödinger equation near the nuclear boundary. We have chosen the Woods-Saxon potential for the nuclear field. The value of the half-life of $^{214}_{84}$Po calculated from this penetration factor comes out to be twice the experimental value.

INTRODUCTION

The various calculations of the penetration factor in the process of α-decay that have been made till recently according to one-body model, give different results which indicate that there is room for finding out a reasonably accurate approximation method. As for the potential for such calculations we have used the Woods—Saxons' diffuse potential model near the nuclear surface, since we feel it approaches reality more than others; outside the nuclear potential Coulomb field predominates. The calculation of the wave function in the region where only Coulomb potential is present has been made by the Riccati method as given by Abramowitz (1949) (cf. Froberg, 1955). So we get an accurate solution of the Schrödinger equation for the Coulomb region. In solving the Schrödinger equation in the neighbourhood of the nuclear boundary we have employed, instead of the WKB approximation, the method of solution given by Lanczos (1938), according to which the differential equation has been equated to an error term which is proportional to Tshebysheff's polynomial of a given order. Consequently the equation admits of a finite power series solution. The error term vanishes at the zero points of the Tshebysheff's polynomial and thus an interpolating power series solution has been obtained. In this case the error term is of an oscillatory character, and the maximum error at any point of the range is less than that of the Taylor's series solution with the same number of terms.

The penetration factor has been calculated from the value of the wave function at the point near the nuclear boundary where the potential energy is equal to the kinetic energy of the emitted α-particle ($r = r_2$).

We have next estimated the half-life from the calculated value of the penetration factor.

MATHEMATICAL FORMULATION

The equation for u, which is r times the wave function of the radial part of the Schrödinger equation can be written as

$$\frac{d^2u}{dr^2}+\frac{2m}{\hbar^2}[E-W(r)]u=0 \qquad \ldots \quad (1)$$

where $W(r)$ for $l=0$ takes the form as

$$W(r)=U(r)+V(r).$$

where $$U(r)=\frac{2(Z-2)e^2}{r} \quad \text{for} \quad r>r_1$$

$$=U(r_1)=\text{constant} \quad \text{for} \quad r<r_1,$$

and $$V(r)=-\frac{V_0}{1+e^{\frac{r-R}{a}}} \quad \text{for} \quad r<r_1$$

$$=0, \quad \text{for} \quad r>r_1.$$

r_1 is taken as the point where the nuclear potential drops to $-\frac{V_0}{100}$. For $r<r_1$, the Coulomb potential is assumed to have a constant value.

To solve equation (1) in the region $R\leqslant r\leqslant r_1$ we write it in the form

$$\frac{d^2u}{dx^2}+\left\{\frac{\lambda^2}{1+\beta e^x}-k^2\right\}u=0 \qquad \ldots \quad (2)$$

where, $$x=\frac{r}{a}, \qquad \lambda^2=\frac{2m}{\hbar^2}V_0a^2$$

$$\beta=e^{-R/a}, \; k^2=\frac{2m}{\hbar^2}(U-E)a^2$$

we now seek solutions of the above differential equation of the form

$$u\sim e^{\mp kx}\,.\,F_{\mp}$$

Substituting $z=e^{-x}$, we get,

$$z(z+\beta)\frac{d^2F}{dz^2}+(z+\beta)(1\pm 2k)\frac{dF}{dz}+\lambda^2F=0 \qquad \ldots \quad (3)$$

For our later calculations the independent variable occurring as the argument of the Tshebysheff's polynomial has to be normalised such that is varies from zero to one; so we make the transformation

$$p = \frac{z - z_1}{z_2 - z_1}.$$

We get from equation (3)

$$(p+\mu)(p+\nu)\frac{d^2F}{dp^2} + (p+\nu)(1\pm 2k)\frac{dF}{dp} + \lambda^2 F = 0, \qquad \dots (4)$$

where $\mu = \dfrac{z_1}{z_2 - z_1}$ and $\nu = \dfrac{z_1+\beta}{z_2 - z_1}$.

we now replace F by a polynomial of finite degree n and following Lanczos equate the differential equation to a term proportional to Tshebysheff's polynomial of order n. We write

$$D(F) = \tau T_n(p)$$

where $D \equiv (p+\mu)(p+\nu)\dfrac{d^2}{dp^2} + (p+\nu)(1\pm 2k)\dfrac{d}{dp} + \lambda^2$

Let $F = \sum_{i=0}^{n} a_i p^i$ where $a_0 = 1$.

$$T_n = \sum_{i=0}^{n} B_i p^i \quad \text{(for values of } B \text{ vide Lanczos p. 140).}$$

Substituting the above expressions for F and T in the differential equation (4) we get the recursion formulae by comparing the coefficients of the same power of p on both sides of the equation.

$$\tau B_r = a_{r+2}[\mu\nu(r+1)(r+2)] + a_{r+1}[(\mu+\nu)r(r+1) + \delta\nu(r+1)] + a_r[r(r-1) + \delta r + \lambda^2].$$

$$\tau B_n = a_n[\lambda^2 + \delta n + n(n-1)]$$

$$\tau B_{n-1} = a_n[n(n-1)(\mu+\nu) + n\delta\nu] + a_{n-1}[(n-1)(n-2) + (n-1)\delta + \lambda^2].$$

where $\delta = \begin{cases} 1+2k \\ 1-2k \end{cases}$

Therefore the solution of the differential equation is known from the recursion formulae except for an arbitrary constant multiplier. The two values of δ give two solutions. We write therefore for the solution near the surface of the nucleus $(r < r_1)$

$$u = Ae^{-kx}F_{-}(p)+Be^{+kx}F_{+}(p). \qquad \ldots \quad (5)$$

Now for the outside region $(r > r_1)$ only Coulomb potential is effective. We can write for this region,

$$\frac{d^2u}{d\rho^2}+\left(1-\frac{2\eta}{\rho}\right)u = 0, \qquad \ldots \quad (6)$$

where $$\rho = \alpha r = \sqrt{\frac{2mE}{\hbar^2}}\,.\,r$$

$$2\eta = \frac{2m}{\hbar^2}\,.\,\frac{2(Z-2)e^2}{\alpha}$$

We must seek the appropriate solution of the above equation (6) near the boundary $(r = r_1)$ where the inner solution $(r < r_1)$ is to be matched with the outer solution $(r > r_1)$.

The equation has solutions F_0, G_0 the asymptotic behaviour of which is given by

$$F_0 \sim \sin\theta$$

$$G_0 \sim \cos\theta,$$

as $$\rho \to \infty,$$

where $\theta = \rho-\eta\log 2\rho+\sigma.$ and $\sigma = \arg\,\Gamma(i\eta+1)$.

The combination G_0+iF_0 will satisfy our boundary condition that at infinity the α-particle should behave as a free outgoing particle. Now for different ranges defined by values of ρ and η, different representations of F_0 and G_0 are given. In our region $(\rho < 2\eta)$ we take the representation of F_0 and G_0 given by Abramowitz, based on Riccati's method as quoted by C. G. Froberg (1955),

$$F_0 = \tfrac{1}{2}e^{\varphi(t,\eta)}.$$

$$G_0 = e^{\psi(t,\eta)},$$

where $$t = \frac{\rho}{2\eta}$$

$$Q(t,\eta) = 2\eta g_0+g_1+(2\eta)^{-1}g_2+(2\eta)^{-2}g_3+\ldots\ldots\ldots\ldots\ldots$$

$$\psi(t,\eta) = -2\eta g_0+g_1-(2\eta)^{-1}g_2+(2\eta)^2g_3-\ldots\ldots\ldots\ldots$$

From continuity of u and $\frac{du}{dr}$ at the point $r = r_1$, we fix the constants A and B of equation (5). It is found that F_0 and $\frac{dF_0}{dr}$ are negligible in comparison with G_0 and $\frac{dG_0}{dr}$ at $r = r_1$. The values of u at $r = r_2$ where $R < r_2 < r_1$, is next calculated from equation (5) and the penetration factor as defined by Blatt and Weisskopf (1954) is as follows

$$P = \frac{1}{|u(r_2)|^2}$$

To the probability of penetration per second we write

$$\lambda = nP$$

where n is the number of times the α-particle hits the barrier wall. If the α-particle moves with a velocity v within the crater of the nucleus of radius R, then $n = \frac{v}{2R}$, further we take the de Broglie wavelength to be equal to $2R$, we obtain (Max Born, 1951).

$$n = \frac{h}{4mR^2}$$

Now the half-life can be calculated from the expression

$$T = \frac{\log_e 2}{\lambda} = \frac{0.6931}{\lambda}.$$

RESULT AND DISCUSSION

The numerical calculations are made for Polonium (RaC′) with $A = 214$, $Z = 84$. The values of parameters are the same as used by Igo and Thaler (1957).

$$R = 1.35A^{1/3}+1.3 \text{ in Fermi} = 8.32\times10^{-13} \text{ cm}$$

$$a = 0.5\times10^{-13} \text{ cm}, \qquad V_0 = 45 \text{ Mev}.$$

$$m = 6.52\times10^{-24} gm.$$

$$E = 7.714 \text{ Mev}.$$

We have found the value of $P = .059\times10^{-16}$. With the same values of the parameters and applying WKB method Rassmussen (1959) found the value of α-emission width $\delta^2 \sim \cdot15$, from which the penetration factor P comes out to be $\sim 1.17\times 10^{-16}$. Our value is thus about 20 times smaller than this. It is worth while to mention here that we have taken $U(r)$ as constant equal to $U(r_1)$ for the inner solutions whereas the actual Coulomb contribution for that part would be somewhat larger. Consequently the value of P is expected to be less than the value we have obtained. With our calculated value of the penetration factor the half-life comes out to be

$$T = 3.201 \times 10^{-4} \text{ sec.}$$

which is about twice the experimental value (Rasmussen, 1959)

$$T = 1.636 \times 10^{-4} \text{ sec.}$$

In view of the uncertainty of the values of the parameters, this agreement may be considered quite satisfactory.

The accuracy of WKB method has often been doubted (Blatt, and Weisskopf, 1954). That is why we have not used it and tried Lanczos' method. The result obtained here gives appreciably better agreement than that given by the WKB method.

The application of this technique to the excited states ($l \neq 0$) is under progress.

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